$$W = c[u]. \tag{4.3}$$

Using the relation (4.2) and the results in [5], one finds that the formula (4.3) for an additively compressible medium assumes the form

$$W = (c\cos\varphi - \sigma\sin\varphi)\{[u]^2 + [v]^2\}^{1/2} + \sigma[v].$$

By employing theorems on extremals the latter expression can be used to find upper bounds for limit loads in the case of motion of granular materials subjected to high pressures.

LITERATURE CITED

1. D. Drucker and W. Prager, "Soil mechanics and plastic analysis or limit design," Quart. Appl. Math., 10, No. 2 (1952).
2. R. Shild, "Mixed boundary value problems in soil mechanics," Quart. Appl. Math., 11, No. 1 (1953).
3. D. D. Ivlev, "Theory of compressible and ideally plastic media," Prikl. Mat. Mekh., 27, No. 3 (1963).
4. V. V. Sokolevskii, Statics of Granular Media [in Russian], Fizmatgiz, Moscow (1960).
5. I. S. Degtyarev, "Extremal theorems and discontinuous solutions in the case of additively compressible plastic body," in: Mechanics of a Deformable Solid Body [in Russian], No. 1, Izd. Kuibyshevsk. Univ. (1975).

DYNAMIC DEFLECTION OF STIFFLY PLASTIC RESTRAINED CIRCULAR PLATES WITH THE EFFECTS OF SHEAR AND ROTATIONAL INERTIA TAKEN INTO ACCOUNT

Yu. V. Nemirovskii and A. R. Skovoroda

UDC 539.38.384.2+539.38.386

Upon the action of transient dynamic loads of high intensity, a large part of the internal energy supplied to a structure can be dissipated into the work of plastic deformations prior to the structure being destroyed or receiving inadmissible residual deformations. Approaches based on the model of a stiffly plastic body have found wide application for the solution of the corresponding problems based on estimation of the extent of damage to structures from the action of "explosive" loads. A detailed review of Soviet and foreign research conducted in recent years in this area is given in [1]. Experimental investigations conducted in a number of papers [2-4] have revealed that the values of the residual deflections and rotation angles measured in the experiments turn out to be appreciably less than the theoretical values, amounting to approximately 20-80% of them. This discrepancy is explained by the effect of a number of factors which are not taken into account in the theory mentioned above. In particular, the effect of rotational inertia and the limitedness of resistance to transverse shears is not taken into consideration in all investigations known up to now. A theory of the dynamic deflection of circular plates made of a stiffly plastic material is developed in this paper which takes account of rotational inertia and the reduced resistance to transverse shear. It is shown that the nature of the dynamic behavior and the energy dissipation mechanism in the case of plastic deformations is significantly different in this case than in the theories mentioned above.

§1. We will consider the axisymmetrical deformation of a circular plate made of an ideally stiffly plastic material. We will assume the following kinematic hypotheses of the deformation of the plate (in the cylindrical coordinate system r, φ, z tied to the median surface):

$$u_r = zu(r,t),\ u_\varphi \equiv 0,\ u_z = W(r,t),$$

where u_r, u_φ, and u_z are the components of the displacement vector and t is the time. Using Lagrange's variational principle in combination with the d'Alembert principle, we obtain, respectively, the equations of motion of the plate

$$(1/3)\ddot{u} = m_1' + x^{-1}(m_1 - m_2) - ql + \varphi,\quad \ddot{w} = q' + x^{-1}q + p \tag{1.1}$$

Novosibirsk, Barnaul. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 124-133, March-April, 1978. Original article submitted December 17, 1976.

and the boundary conditions at the edge $x = x_j$

$$\left[\delta u\left(m_1 n_j - l^{-1}\int_{-1}^{1}\varphi_j\zeta d\zeta\right)\right]\Big|_{x=x_j} = 0,\quad \left[\delta w\left(q n_j - l^{-1}\int_{-1}^{1}p_j d\zeta\right)\right]\Big|_{x=x_j} = 0,$$

where

$$m_i = \int_{-1}^{1}\sigma_i\zeta d\zeta \quad (i = 1, 2);\quad q = \int_{-1}^{1}\tau_{13}d\zeta;$$
$$p = (P_z^+ + P_z^-)R(\sigma_0 H)^{-1};\quad \varphi = (P_r^+ - P_r^-)R(\sigma_0 H)^{-1};$$
$$\zeta = zH^{-1};\ x = rR^{-1};\ l = RH^{-1};\ w = WH^{-1};$$
$$\sigma_1 = \sigma_r\sigma_0^{-1};\quad \sigma_2 = \sigma_\varphi\sigma_0^{-1};\quad \tau_{13} = \sigma_{rz}\sigma_0^{-1};$$
$$\varphi_j = RP_r(r_j)(\sigma_0 H)^{-1};\ p_j = RP_z(r_j)(\sigma_0 H)^{-1};$$
$$\tau^2 = \sigma_0 t^2(2\rho HR)^{-1}.$$

Here m_1, m_2 are the dimensionless radial and circumferential bending moments; q is the dimensionless intersecting force, 2H is the plate thickness; R is its radius; $P_z^\pm$, $P_r^\pm$ are the components of the surface load at the surfaces $z = \pm H$ of the plate; $P_r(r_j)$, $P_z(r_j)$ are the components of the load at the edge $r = r_j$; σ_r, σ_φ, σ_{rz} are the stress components; σ_0 is the yield stress of the plate material; ρ is the specific density; and n_j is the direction cosine of a small area on the contour x_j. A derivative with respect to the dimensionless coordinate x is denoted by a prime, a derivative with respect to the dimensionless time τ is denoted by a dot, and δ denotes the sign of the variation.

Let us assume that the plate material is ideal stiffly plastic material which satisfies in the space m_1, m_2, q the plasticity condition in the form of the Tresca–St. Venant prism (Fig. 1) and the plastic flow law associated with it,

$$\dot{\varkappa}_i = \sum_{k=1}^{n}\lambda_k\frac{\partial f_k}{\partial m_i},\quad \dot{\varepsilon}_{13} = \sum_{k=1}^{n}\lambda_k\frac{\partial f_k}{\partial q}\qquad (i = 1, 2,\ n = 1-8), \tag{1.2}$$

where

$$\begin{cases}\dot{\varepsilon}_i = \zeta\dot{\varkappa}_i;\quad \dot{\varepsilon}_{13} = \dot{u}(x,\tau) + l^{-1}\dot{w}'(x,\tau);\\ \dot{\varkappa}_1 = l^{-1}\dot{u}';\quad \dot{\varkappa}_2 = \dot{u}(lx)^{-1};\end{cases} \tag{1.3}$$

$$\lambda_k > 0,\quad \text{if}\quad f_k = 1,\ df_k = 0;$$
$$\lambda_k = 0,\ \text{if}\quad f_k < 1,\quad \text{or}\quad f_k = 1, \text{but}\ df_k < 0,$$

and the equations

$$f_k = a_k m_1 + b_k m_2 + c_k q = 1 \quad (k = 1, 2, \ldots, n)$$

determine the equations of the faces of the yield surface. The prism used can serve as a sufficiently satisfactory approximation in m_1, m_2, q space to the plasticity conditions, which are known from [5-8]. The parameters a_k, b_k, and c_k are selected from the condition that the prism length be equal to twice the load of the shear resistance of the plate, and the Tresca hexagon lies at the base of the prism in m_1, m_2 space [4, 9].

At the instant $\tau = \tau_0$ we have the initial conditions

$$\dot{u}(x,\tau_0) = \dot{u}^0(x),\ u(x,\tau_0) = u^0(x),\ \dot{w}(x,\tau_0) = \dot{w}^0(x),\ w(x,\tau_0) = w^0(x). \tag{1.4}$$

The continuity conditions

$$[\dot{u}] = [\dot{w}] = [m_1] = [q] = 0 \tag{1.5}$$

should be satisfied on the plate boundaries separating the plastic modes and corresponding to the various faces or edges of the yield surface used.

§2. In the case of a restrained plate under the action of an impulse uniformly distributed over its surface, the boundary conditions are of the form [9]

$$q(0,\tau) = 0,\ m_1(0,\tau) = -1, \tag{2.1}$$
$$m_1(1,\tau) = 1,\ \dot{w}(1,\tau) = 0.$$

Since we also have $m_2(0, \tau) = -1$ due to symmetry, and $q(x, \tau)$ is an increasing function of x (at least in the vicinity of the center) for the type of loading under discussion, the trajectory of the plastic states on the yield surface has the shape of the curve $abcd$ in Fig. 1. In accordance with this, in the general case plastic modes corresponding to the 3, 9 edge [in the region $0 \le x \le a(\tau)$], the 2, 3, 9, 8 plane [in the region $a(\tau) \le x \le b(\tau)$], and the 1, 2, 8, 7 plane [in the region $b(\tau) \le x \le 1$] are realized in the plate. Then in accordance with (1.1)-(1.3) we have

$$\begin{cases} f_1 \equiv -m_1(x,\tau) = 1, \quad f_2 \equiv -m_2(x,\tau) = 1, \\ \ddot{u}(x,\tau) = -3lq(x,\tau), \quad \ddot{w} = p(\tau) + x^{-1}q(x,\tau) + q'(x,\tau); \end{cases} \tag{2.2}$$

$$q'' + x^{-1}q' - q(x^{-2} + \varepsilon^2) = 0, \ \varepsilon^2 = 3l^2 \tag{2.3}$$

in the region $0 \le x \le a(\tau)$. Consequently, taking (1.4) and (2.1) into account, we obtain

$$q(x,\tau) = C_1(\tau)I_1(\varepsilon x); \tag{2.4}$$

$$\begin{cases} \dot{u}(x,\tau) = -3lC_{11}I_1(\varepsilon x) + \dot{u}^0(x), \\ \dot{w}(x,\tau) = I + \varepsilon C_{11}I_0(\varepsilon x) + \dot{w}^0(x); \end{cases} \tag{2.5}$$

$$\begin{cases} u(x,\tau) = -3lC_{12}I_1(\varepsilon x) + (\tau-\tau_0)\dot{u}^0(x) + u^0(x), \\ w(x,\tau) = J + \varepsilon C_{12}I_0(\varepsilon x) + (\tau-\tau_0)\dot{w}^0(x) + w^0(x), \end{cases} \tag{2.6}$$

$$I = \int_{\tau_0}^{\tau} p(\tau)\,d\tau, \quad J = \int_{\tau_0}^{\tau} I(\tau)\,d\tau, \quad C_{11} = \int_{\tau_0}^{\tau} C_1(\tau)\,d\tau,$$

$$C_{12} = \int_{\tau_0}^{\tau} C_{11}(\tau)\,d\tau, \quad I_0(y) = \frac{dI_1(y)}{dy} + y^{-1}I_1(y),$$

where $I_1(y)$ is the Bessel function of imaginary argument. In accordance with the plasticity condition and the plastic flow law (1.2), the solution under discussion for the 9, 3 mode in Fig. 1 is realized upon satisfaction of the inequalities

$$-\alpha \leqslant q(x,\tau) \leqslant \alpha, \ \dot{u}'(x,\tau) \leqslant 0, \ \dot{u}(x,\tau) \leqslant 0 \\ (0 \leqslant x \leqslant a(\tau), \ \tau \geqslant \tau_0). \tag{2.7}$$

In the region $a(\tau) \le x \le b(\tau)$ we have from the equations of motion, the plasticity condition, and the plastic flow law (for the 3, 2, 8, 9 face in Fig. 1)

$$m_2 = -1, \ \dot{\varepsilon}_{13} = 0, \ \dot{\varkappa}_1 = 0, \ \dot{\varkappa}_2 = 0; \tag{2.8}$$

$$\dot{u}(x,\tau) = f(a,\tau), \quad \dot{w}(x,\tau) = -lf(a,\tau)(x-a) + g(a,\tau); \tag{2.9}$$

$$\begin{cases} q(x,\tau) = (6x)^{-1}\{6a(\tau)C_1(\tau)I_1(\varepsilon x) + \\ + (x-a)[3(x+a)(\dot{g} + l\dot{a}f - p) - l(x-a)(2x+a)\dot{f}]\}, \\ m_1(x,\tau) = (12x)^{-1}\{2[(x^2-a^2)\dot{f} + 6(x-a)alC_1I_1(a\varepsilon) - 6x] \\ + l(x-a)^2[2(x+2a)(\dot{g} + l\dot{a}f - p) - l(x^2-a^2)\dot{f}]\}, \end{cases} \tag{2.10}$$

where $f(a,\tau) = \dot{u}^0(a) - 3lC_{11}I_1(a\varepsilon)$;

$$\dot{f}(a,\tau) = \dot{a}\partial\dot{u}^0(a)/\partial a - 3lC_1I_1(a\varepsilon) - 3l\varepsilon\dot{a}C_{11}[I_0(a\varepsilon) - (a\varepsilon)^{-1}I_1(a\varepsilon)];$$

$$g(a,\tau) = I + \dot{w}^0(a) + \varepsilon C_{11}I_0(a\varepsilon);$$

$$\dot{g}(a,\tau) = p + \dot{a}\partial\dot{w}^0(a)/\partial a + \varepsilon C_1I_0(a\varepsilon) + \varepsilon^2C_{11}\dot{a}I_1(a\varepsilon).$$

Upon integration the equation $m_1(a, \tau) = -1$ was taken into account due to the continuity conditions (1.5) at the boundary $x = a$. The plastic mode (3, 2, 8, 9 in Fig. 1) is valid upon satisfaction of the inequalities

$$\dot{u}(x,\tau) \leqslant 0, \ 0 \geqslant m_1(x,\tau) \geqslant -1 \\ (a \leqslant x \leqslant b, \ \tau \geqslant \tau_0). \tag{2.11}$$

A plastic state corresponding to the 1, 2, 8, 7 plane in Fig. 1 is realized in the region $b(\tau) \le x \le 1$. For this state we have

$$m_1 - m_2 = 1, \ \dot{\varepsilon}_{13} = 0, \ \dot{\varkappa}_1 + \dot{\varkappa}_2 = 0; \tag{2.12}$$

$$\dot{\varkappa}_1 \geqslant 0,\ \dot{\varkappa}_2 \leqslant 0,\ 0 \leqslant m_1(x, \tau) \leqslant 1; \tag{2.13}$$

$$\begin{cases} \dot{u}(x, \tau) = b(\tau) f(a, \tau) x^{-1}, \\ \dot{w}(x, \tau) = -lb(\tau) f(a, \tau) \ln(xb^{-1}) + g(a, \tau) - l(b-a) f(a, \tau); \end{cases} \tag{2.14}$$

$$\begin{cases} q(x, \tau) = (4x)^{-1}\{4bq(b, \tau) + 2(x^2 - b^2)(\dot{F} - p) - \\ - l(\dot{b}f + b\dot{f})[2(x^2 \ln x - b^2 \ln b) - x^2 + b^2]\}, \\ 12m_1(x, \tau) = (\dot{b}f + \dot{f}b)\{[(2\ln b - 1)b^2\varepsilon^2 + 4]\ln(xb^{-1}) + \\ + \varepsilon^2(x^2 - b^2) - \varepsilon^2(x^2 \ln x - b^2 \ln b)\} + 12[blq(b, \tau) - 1] \times \\ \times \ln(xb^{-1}) + 3l(\dot{F} - p)[2b^2 \ln(bx^{-1}) + x^2 - b^2], \end{cases} \tag{2.15}$$

where

$$\dot{F}(\tau) = \dot{g}(a, \tau) + l\dot{a}f - l(b - a)\dot{f}(a, \tau) + l(b\dot{f} + \dot{b}f)\ln b. \tag{2.16}$$

At the same time, due to the continuity conditions, it is taken into account during integration that

$$m_1(b, \tau) = 0. \tag{2.17}$$

Since due to the last of the boundary conditions (2.1) we have $\dot{w}(1, \tau) = 0$ for $\tau \geq \tau_0$, then $\ddot{w}(1, \tau) = 0$ when $\tau \geq \tau_0$, i.e., $\dot{F}(\tau) \equiv 0$. Using this equation, Eq. (2.17), the boundary condition $m_1(1, \tau) = 1$, and Eqs. (2.10), (2.14), (2.15), and (2.16) for the determination of the functions $a(\tau)$, $b(\tau)$, and C_{11}, we obtain the following system of differential equations:

$$\begin{gathered} \varphi_1\dot{a} + \psi_1\dot{b} + \omega_1\dot{C}_{11} = d_1, \\ \varphi_2\dot{a} + \omega_2\dot{C}_{11} = d_2,\ \varphi_3\dot{a} + \psi_3\dot{b} + \omega_3\dot{C}_{11} = d_3, \end{gathered} \tag{2.18}$$

where

$$\varphi_1 = -l(b - a - b\ln b)\left[\frac{\partial \dot{u}^0(a)}{\partial a} - 3lC_{11}\frac{\partial I_1(a\varepsilon)}{\partial a}\right];$$

$$\psi_1 = fl\ln b,\quad \omega_1 = \varepsilon[I_0(a\varepsilon) + \varepsilon(b - a - b\ln b)I_1(a\varepsilon)];$$

$$\varphi_2 = l^{-1}(b + a)[2 - l^2(b - a)^2]\left[\frac{\partial \dot{u}^0(a)}{\partial a} - 3lC_{11}\frac{\partial I_1(a\varepsilon)}{\partial a}\right];$$

$$\omega_2 = (b - a)\{2\varepsilon(b + 2a)I_0(a\varepsilon) + [\varepsilon^2(b^2 - a^2) - 6]I_1(a\varepsilon)\};$$

$$36\varphi_3 = \{\varepsilon^2 b(1 - b^2) - 2\ln b[3b\{2 + \varepsilon^2 b^2(\ln b - 1)\} - \varepsilon^2(b - a)^2(2b + a)]\}\left[\frac{\partial \dot{u}^0(a)}{\partial a} - 3lC_{11}\frac{\partial I_1(a\varepsilon)}{\partial a}\right]; \tag{2.19}$$

$$12\psi_3 = f\{\varepsilon^2(1 - b^2) - 2[2 + \varepsilon^2 b^2(\ln b - 1)]\ln b\};$$

$$12\omega_3 = l[\{[12(b - a) + 2\varepsilon^2\{3b^3\ln b - (b - a)^2(2b + a) - 3b^3\}]\ln b - 3\varepsilon^2 b(1 - b^2)\}I_1(a\varepsilon) - 6\varepsilon(b^2 - a^2)I_0(a\varepsilon)\ln b];$$

$$d_1 = -p(\tau);\ d_2 = 12b[l(b - a)]^{-1};$$

$$4d_3 = l(2b^2\ln b + 1 - b^2)p(\tau) - 4(\ln b - 1).$$

Let $\dot{a} = 0$, $\dot{b} = 0$, and let no accelerations $\ddot{u}$, $\ddot{w}$ be present in the region $0 \leq x \leq a$ at some instant of time. One can convince oneself from Eqs. (2.2), (2.5), (2.9), and (2.18) that this situation is possible only when $a = 0$ and

$$p = p_* = -6(lb_*^2)^{-1}, \tag{2.20}$$

where b_* is determined from the equation

$$2b_*^2(1 - \ln b_*) = 3(1 - b_*^2) \qquad (b_* \approx 0.73). \tag{2.21}$$

Equation (2.20) determines the amplitude of the limiting load for a restrained plate [9]. One can verify by using Eqs. (2.2), (2.5), (2.9), (2.14), (2.20), and (2.21) that at the instant of time at which $a = 0$, $b = b_*$, $\dot{a} = 0$, and $d = 0$, we also have $\dot{u} = \ddot{u} = \dot{w} = \ddot{w} = 0$ for all $0 \leq x \leq 1$.

Thus, the motion under discussion can arise only in the case of loads exceeding the limiting value p_*. Taking rotational inertia into account results in a significantly different mechanism of energy dissipation than in the classical solution [9]: Here the plastic zone in the vicinity of the center, for which $m_1(x, \tau) = m_2(x, \tau) = -1$, exists from the start of the motion and until its cessation. This fact results in significantly more intensive energy dissipation during plastic deformations and should lead to an appreciable lowering of the level of residual deflections. The latter are determined by the integration of Eqs. (2.5), (2.9), and (2.14) over time.

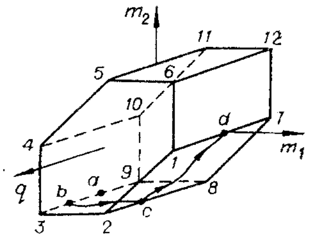

Fig. 1

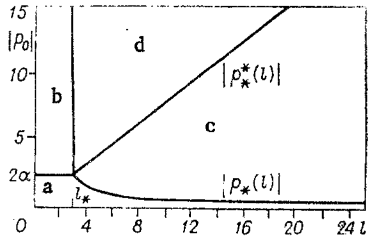

Fig. 2

The maximum residual deflection at the center of the plate is equal to

$$w_f(0,\tau_f)=w^0(0)+\dot{w}^0(0)(\tau_f-\tau_0)+J(\tau_f)+\varepsilon\int_{\tau_0}^{\tau_f}C_{11}(\tau)\,d\tau, \tag{2.22}$$

where τ_f is the stopping time of the plate, which is determinable from the equation $a(\tau_f) = 0$. We obtain the initial conditions $a_0 = a(\tau_0)$ and $b_0 = b(\tau_0)$ necessary for integration of the system (2.18) from (2.18) at $\tau = \tau_0$.

The solution derived in this section occurs upon satisfaction on the segment $0 \le x \le 1$ and during the time $\tau_0 \le \tau \le \tau_f$ of the inequalities (2.7), (2.11), and (2.13). One can verify that all these inequalities are fulfilled except $q(x, \tau) \le \alpha$. The function $q(x, \tau)$ is monotonically increasing with x, in accordance with the derived solution. Then the equation $q(1, \tau_1) = \alpha$ determines when shear of the plate at the support is possible in the case of a nondecreasing load. In this connection the solution discussed above is valid in the time interval $\tau_0 \le \tau \le \tau_2$, where $\tau_2 = \min(\tau_f, \tau_1)$ and $|p(\tau)| \ge |p_*|$. If the load decreases from the initial instant of time $\tau = \tau_0$, then we determine the values of a_0, b_0, $\dot{C}_{11}^0 = \dot{C}_{11}(\tau_0)$, and the dependence of the peak of the shearing load on l, i.e., $p_*^*(l)$, from the system of equations (2.18) at $\tau = \tau_0$ in combination with the equation $q(1, \tau_0) = \alpha$ or the relation

$$2\left[2b_0q(b_0,\tau_0)-(1-b_0^2)\,p(\tau_0)\right]+l(\dot{b}f+\dot{f}b)\big|_{\tau=\tau_0}\left[2b_0^2\ln b_0+1-b_0^2\right]=4\alpha\ . \tag{2.23}$$

Then the solution under discussion is valid upon satisfaction of the inequalities

$$|p_*|\leqslant|p(\tau)|\leqslant|p_*^*(l)|\qquad(\tau_0\leqslant\tau\leqslant\tau_f).$$

In the case $|p_*| = |p(\tau_0)| = |p_*^*|$ we evidently have $\dot{a}(\tau_0) = \dot{b}(\tau_0) = a(\tau_0) = 0$ and $b(\tau_0) = b_*$. Then from (2.23) we obtain

$$|p_*|=2\alpha \text{ and } l=l_*=3\,(\alpha b_*^2)^{-1}.$$

One can verify that $|p_*^*(l)| \ge |p_*|$ if $l \ge l_*$, and $\lim\limits_{l\to\infty}|p_*^*| = \infty$.

§3. If the peak of the (nonincreasing) load exceeds the value $|p_*^*|$, then the motion of the plate occurs with those same plastic modes as in Sec. 2 with shear at the edge. The latter situation results in the fact that in this case one should use the boundary condition $q(1, \tau) = \alpha$ instead of the boundary condition $\dot{w}(1, \tau) = 0$. Then the solution of the problem is again determined by Eqs. (2.2)-(2.6), (2.8)-(2.10), (2.12)-(2.14), (2.18), and (2.22), the only difference being that the coefficients φ_2, ω_2, and d_2 of the system (2.18) maintain their values from (2.19), but one should replace the coefficients φ_1, ψ_1, ω_1, d_1, φ_3, ψ_3, ω_3, and d_3 in this case by the coefficients φ_{11}, ψ_{11}, ω_{11}, d_{11}, φ_{31}, ψ_{31}, ω_{31}, and d_{31}, respectively, where,

$$12\varphi_{11}=l\left[3b(2\ln b+1-b^2)-2(b-a)(3-a^2-ab-b^2)\right]\left[\frac{\partial\dot{u}^0(a)}{\partial a}-3lC_{11}\frac{\partial I_1(a\varepsilon)}{\partial a}\right], \tag{3.1}$$

$$4\psi_{11}=fl(2\ln b+1-b^2),\quad 12\omega_{11}=6\varepsilon(1-a^2)I_0(a\varepsilon)+I_1(a\varepsilon)\{12a\ +$$

$$+\,\varepsilon^2\left[2(b-a)(3-a^2-ab-b^2)-3b(2\ln b+1-b^2)\right]\};\qquad d_{11}=\alpha;$$

$$12\varphi_{31}=12\varphi_3+\varepsilon^2(1-b^2+2b^2\ln b)(b\ln b-b+a)\left[\frac{\partial\dot{u}^0(a)}{\partial a}-3lC_{11}\frac{\partial I_1(a\varepsilon)}{\partial a}\right];$$

$$12\psi_{31}=12\psi_3+\varepsilon^2 f\ln b(1-b^2+2b^2\ln b);$$

$$4\omega_{31}=4\omega_3+l(1-b^2+2b^2\ln b)\left[\varepsilon^2(b-a-b\ln b)I_1(a\varepsilon)+\varepsilon I_0(a\varepsilon)\right];\ d_{31}=1-\ln b.$$

As is evident from these equations, the coefficients and the right-hand sides of Eq. (2.18) do not depend in this case on the form of the load $p(\tau)$, i.e., a variation of the load in the case of motion of a plate with shear at the support is not expressed in the nature of the variations of the plastic states inside the plate but is expressed only in a variation of its displacements and velocities.

We have $l\dot{\varepsilon}_{13} = \dot{w}' + l\dot{u} = 0$ and $\dot{\varkappa}_2 = (lx)^{-1}\dot{u} \le 0$ in the section $0 \le x \le 1$ from the laws of plastic flow. Thus $\dot{w}' \ge 0$ in the entire plate; since at the same time $\dot{w} \le 0$, then evidently the smallest absolute value of the deflection velocity is attained at the support. Thus the stopping time τ_3 of the motion with shear being discussed is determined from the equation $\dot{w}(1, \tau_3) = 0$ or $[g(a, \tau) - lf(a, \tau)(b - a - b\ln b)]_{\tau=\tau_3} = 0$. The subsequent motion ($\tau > \tau_3$) of the plate is completely determined by the relations given in Sec. 2 if one replaces everywhere τ_0 by τ_3 and $\dot{u}^0(x)$, $\dot{w}^0(x)$, $u^0(x)$, and $w^0(x)$ by the quantities $\dot{u}(x, \tau_3)$, $\dot{w}(x, \tau_3)$, $u(x, \tau_3)$, and $w(x, \tau_3)$, respectively. The stopping time τ_4 of this phase of the motion is evidently determined from the equation $a(\tau_4) = 0$. The procedure for solution of the problem is completely identical to that described in Sec. 2.

§4. The case of motion of a stiff plate with shear at the support is possible. In this case $\dot{\varkappa}_1(x, \tau) = \dot{\varkappa}_2(x, \tau) = \dot{\varepsilon}_{13}(x, \tau) = 0$. Thus $\dot{u}(x, \tau) = 0$ and $\dot{w}'(x, \tau) = 0$, i.e., $\dot{w}(x, \tau) = \dot{c}(\tau)$. Then we obtain

$$2q(x, \tau) = x(\ddot{c} - p)$$

from the second equation of motion (1.1) with the boundary condition $q(0, \tau) = 0$ taken into account. Taking account of the equation $q(1, \tau) = \alpha$, we obtain

$$\dot{w} = I + 2\alpha(\tau - \tau_0), \quad w = J + \alpha(\tau - \tau_0)^2 \tag{4.1}$$

in the case of null initial conditions. The stopping time τ_5 is determined from the equation

$$I(\tau_5) + 2\alpha(\tau_5 - \tau_0) = 0,$$

and the maximum residual deflection is equal to

$$w_{\max}(\tau_5) = J(\tau_5) + \alpha(\tau_5 - \tau_0)^2.$$

The motion under discussion occurs when $|p| > |p_{**}|$, where p_{**} is determined from the equation $\ddot{w} \equiv 0$, so that

$$p_{**} = -2\alpha.$$

The condition that the plate remain stiff along the section $0 \le x \le 1$ corresponds to the requirement $|p_{**}| < |p_*|$ or $l \le l_*$. We note that if one sets $\dot{a} = \dot{b} = a = 0$ in the system of equations (2.18), with the coefficient replacements indicated in Sec. 3 by the quantities from (3.1) taken into account, then one obtains $l = l_*$, $b = b_*$, and $\varepsilon\dot{C}_{11} = 2\alpha$ from this system. Substituting these values into Eqs. (2.5), (2.9), and (2.14) with the null initial conditions $\dot{u}^0(x) = 0$, $\dot{w}^0(x) = 0$, $u^0(x) = 0$, and $w^0(x) = 0$, we obtain Eqs. (4.1). Thus the solutions of this and the preceding section are completely joined at $l = l_*$.

Thus, in the general case four states are possible for a restrained plate uniformly loaded by a distributed impulse which are indicated systematically in Fig. 2 [a) fixed stiff plate; b) the motion of a stiff plate with shear at the support; c) motion of a deflected plate without shear at the support; d) motion of a deflected plate with shear at the support] as a function of the stiffness parameter l of the plate and the value of the peak load.

§5. The solution expounded above for a restrained plate will occur for a hinge-supported plate with the boundary condition $m_1(1, \tau) = 0$ if one discards all the formulas on the segment $b \le x \le 1$ but assumes $b \equiv 1$ for the formulas on the segment $a \le x \le b$ and sets $\dot{w}(b, \tau) = \dot{w}(1, \tau) = 0$ in the absence of shear and $q(b, \tau) = q(1, \tau) = \alpha$ or in the case of motion with shear. It is necessary in the system (2.18) to discard the third equation. In addition, one should also set $b_* = 1$ in the expression for l_*. For this case the curves illustrated in Fig. 2 are exact for the parameter value $\alpha = 2/\sqrt{3}$. Taking into account what has been said, we will cite here only the formulas which determine the residual deflection of the plate in the case of motion without shear with null initial data:

$$\begin{aligned} w(x, \tau_f) &= J(\tau_f) + \varepsilon \int_0^{\tau_f} A(x, \tau) A_1 I(\tau)\, d\tau \quad \text{for} \quad x \geqslant a_0 \quad \text{and} \\ w(x, \tau_f) &= J(\tau_f) + \varepsilon_0 I_0(x\varepsilon) \int_0^{\tau_6} C_1^0(\tau)\, d\tau + \varepsilon \int_{\tau_6}^{\tau_f} A(x, \tau) A_1 I(\tau)\, d\tau \quad \text{for} \quad x \leqslant a_0. \end{aligned} \tag{5.1}$$

Here

$$A(x, \tau) = I_0(a\varepsilon) + \varepsilon(x - a) I_1(a\varepsilon); \; A_1 = A^{-1}(1, \tau);$$

τ_f is the stopping time, which is determinable from the equation $a(\tau_f) = 0$, τ_6 is determined from the equation $a(\tau_6) = x$, and $a(\tau)$ is found from the equation

$$I\varphi\dot{a} = d, \tag{5.2}$$

where

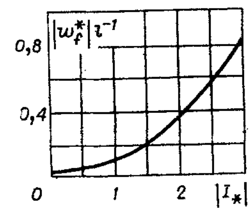

Fig. 3

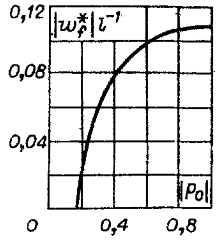

Fig. 4

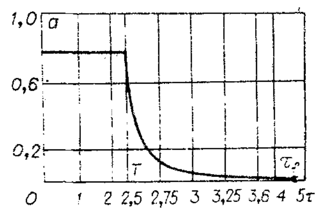

Fig. 5

$$\varphi = \{I_0(a\varepsilon)[\varepsilon^2(1-a)^2\,(3a+1)+6(1+a)]+12a\varepsilon(1-a)I_1(a\varepsilon)\}[I_0(a\varepsilon)-(a\varepsilon)^{-1}I_1(a\varepsilon)];$$

$$d = A_1[12l^{-1}(1-a)^{-1}A_1+p\varepsilon^{-1}(1-a)\{2\varepsilon(1+2a)I_0(a\varepsilon)+[\varepsilon^2(1-a^2)-6]I_1(a\varepsilon)\}].$$

The initial value $a_0 = a(0)$ is determined from the equation

$$12A_1(a_0)\,\varepsilon l^{-1}+p_0(1-a_0)^2\{2\varepsilon(1+2a_0)\,I_0(a_0\varepsilon)+[\varepsilon^2(1-a_0^2)-6]\,I_1(a_0\varepsilon)\}=0 \tag{5.3}$$

$$(p_0 = p(0)),$$

whence it is evident that $a_0 \to 1$ as $p_0 \to -\infty$ or $l \to \infty$ and $a_0 \to 0$ as $p_0 \to p_* = -6l^{-1}$.

Equations (5.1)-(5.3) occur for a nonincreasing load if the amplitude of the load satisfies the inequality

$$p_* > p_0 > p_*^*,$$

where p_*^* is determined from the equation

$$p_*^*[6a_0I_1(a_0\varepsilon)+3(1-a_0^2)\,\varepsilon I_0(a_0\varepsilon)+\varepsilon^2(1-a_0)^2(2+a_0)\,I_1(a_0\varepsilon)]+6\alpha\varepsilon A_1(a_0)=0. \tag{5.4}$$

One can see that $p_*^* \to -\infty$ as $l \to \infty$ and $p_*^* \to p_*$ as $l \to l_* = 3\alpha^{-1}$.

Experimental data are given in [4] on the dependence of the residual deflections on the size of the explosive loading impulse for a hinge-supported plate. The experiments were performed on plates made of Al 6061-T6 aluminum alloy with the parameters $\sigma_0 = 28{,}958 \cdot 10^4$ N/m², $\rho = 2704$ kg/m³, R = 10.16 cm, and H = 0.318 cm and of CRSteel 1018 steel with the parameters $\sigma_0 = 27{,}234 \cdot 10^5$ N/m², $\rho = 7829$ kg/m³, R = 10.16 cm, and H = 0.315 cm. It is noted that the impulse was practically of rectangular shape. In connection with this fact let us adopt the approximation

$$p(\tau) = \begin{cases} p_0, & \tau < T, \\ p_0 + k(\tau - T), & T \leqslant \tau \leqslant T - p_0k^{-1}, \\ 0, & \tau > T - p_0k^{-1} \end{cases} \tag{5.5}$$

for $p(\tau)$ in connection with the calculations based on Eqs. (5.1)-(5.4). In this connection the total impulse of the load is equal to

$$I_* = p_0T - p_0^2(2k)^{-1}.$$

Equation (5.2) with the initial condition $a(0) = a_0$, which is determinable from (5.3), is solved numerically by the fourth-order Runge–Kutta method. The value of $\dot{a}(0)$ necessary in the calculation is determined from the formula

$$\dot{a}(0) = \dot{p}(0)\,A_1(a_0)(1-a_0)\{\varepsilon(1+2a_0)\,I_0(a_0\varepsilon)+\left[\tfrac{1}{2}\varepsilon^2(1-a_0^2)-3\right]I_1(a_0\varepsilon)\}\left[p_0\frac{\partial I_1(a_0\varepsilon)}{\partial(a_0\varepsilon)}\varepsilon\varphi(a_0)\right]^{-1}$$

as one can convince oneself from (5.2) by applying L'Hospital's rule. We have $\dot{a}(0) = 0$ for a load of the type under discussion. Some results of the calculations for the plate made of Al 6061-T6 alloy with the parameter values indicated above and for k = 300 are given in Figs. 3-5. The dependence of the maximum dimensionless residual deflection w_f^* at the center of the plate on the impulse size I_* is illustrated in Fig. 3 for a fixed value of the load amplitude $p_0 = -0.5$. The dependence of w_f^* on the load amplitude p_0 is illustrated in Fig. 4 for a fixed value of the impulse $I_* = -1$. As is evident from this plot, the residual deflection of the plate depends significantly not only on the impulse size, but also on the size of the load amplitude. Therefore, direct comparison of the results obtained from the numerical calculation with the experimental data from [4] is impossible, since only the values of the impulses are given in [4] and the load amplitude values are absent. The dependence $a(\tau)$ is illustrated in Fig. 5 for the values $I_* = -1$ and $p_0 = -0.4$, and it is evident that the size of the region corresponding to the plastic mode 3, 9 in Fig. 1 remains unchanged prior to when τ = T (while the load level is maintained), sharply decreasing with a decline in the load. This region disappears only at the instant the plate stops.

LITERATURE CITED

1. V. N. Mazalov and Yu. V. Nemirovskii, "The dynamics of thin-walled plastic structures," in: Mechanics, New Developments in Foreign Science, No. 5, Problems of the Dynamics of Elastoplastic Media [Russian translation], Mir, Moscow (1975).
2. E. W. Parkes, "Some simple experiments on the dynamic plastic behaviour of mild-steel beams," Brit. Weld. J., 3, No. 8, 36 (1956).
3. I. J. Mentel, "The plastic deformation due to impact of a cantilever beam with an attached tip mass," J. Appl. Mech., 25, No. 4, 515-524 (1958).
4. A. L. Florence, "Circular plate under a uniformly distributed impulse," Int. J. Solids Struct., 2, No. 1, 37-47 (1966).
5. A. Sawczuk and M. Duszek, "A note on the interaction of shear and bending in plastic plates," Arch. Mech. Stosow., 15, No. 3, 411-426 (1963).
6. N. P. Zhuk and O. N. Shablii, "The limiting equilibrium of shells of revolution and circular plates with shear stresses taken into account," Prikl. Mekh., 8, No. 7 (1972).
7. N. P. Zhuk and O. N. Shablii, "The limiting equilibrium of a circular plate with the shear stresses taken into account," Prikl. Mekh., 9, No. 6 (1973).
8. G. S. Shapiro, "The yield surfaces for ideally plastic shells," in: Problems of the Mechanics of a Continuous Medium [in Russian], Izd. Akad. Nauk SSSR, Moscow (1961).
9. W. Wang and G. Hopkins, "The plastic deformation of a circular plate fixed along the edge under the action of an impulse load," in: Mekhanika [Periodic Collection of Translations of Foreign Articles], No. 3 (1955).

ELASTOPLASTIC STRAIN OF THIN PLATES AND SHELLS UNDER LINEAR HARDENING AND AN IDEAL BAUSCHINGER EFFECT

G. V. Ivanov

UDC 539.3

The elastoplastic strain of thin plates and shells is considered in the case when the elongation and shears are small compared with unity, the hardening is linear, the Bauschinger effect is ideal, and the stresses and strains are related by equations [1, 2]. In solving problems numerically by using the equations [1, 2], it is necessary to evaluate integrals over the plate (shell) thickness and thus to store and process, respectively, information about the stresses, the residual microstresses, and the nature of the strain at the sites over the plate (shell) thickness during the solution. Analogously to the case of ideal elastoplastic strain of plates and shells [3], approximate equations which contain no stresses and relate the strain directly to the forces and moments are formulated below in correspondence to the equations in [1, 2]. The need to evaluate integrals over the plate (shell) thickness drops out in solving problems by using these equations, which simplifies the solution. Numerical experiments performed for a number of strain paths of the shell element exhibit satisfactory agreement of the approximate equations with the equations of [1, 2].

§1. Let us use a Lagrange coordinate system, orthogonal in the unstrained state, to write the equations. For small elongations and shears, the system under consideration can be considered orthogonal in the strain state as well. The strain and stress tensor components are related in the case of elastoplastic strain with linear hardening and an ideal Bauschinger effect (Fig. 1) by the equations [1, 2]

$$\dot{e}_{ij} = (1+\nu)\dot{\sigma}_{ij} - 3\nu\delta_{ij}\dot{\sigma} + \gamma\dot{\eta}_{ij}, \quad \dot{\eta}_{ij} = \lambda s'_{ij};$$
$$\lambda = 0, \quad \text{if} \quad 3T^2 < 1 \quad \text{or} \quad 3T^2 = 1, \quad \dot{T} < 0;$$
$$\lambda > 0, \quad \text{if} \quad 3T^2 = 1, \quad \dot{T} = 0;$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 134-139, March-April, 1978. Original article submitted February 8, 1977.

0021-8944/78/1902-0246$07.50